QR-Decomposition and Diagonalization Linear Algebra X. Du

- Let A be an $m \times n$ matrix. Then A has linearly independent column vectors iff $A^T A$ is invertible.
- **QR-Decomposition**: Let A be an $m \times n$ matrix with linearly independent column vectors. Then $A = QR$, where Q is an orthogonal matrix (i.e. matrix with orthonormal column vectors), and R is an $n \times n$ invertible upper triangular matrix.
    - Find Q by performing the Gram-Schmidt process on the columns vectors of A before normalizing each vector.
    - Find R: Suppose that $\vec{v}_1, \vec{v}_2, ..., \vec{v}_n$ are the column vectors of A and $\vec{u}_1, \vec{u}_2, ..., \vec{u}_n$ are the column vectors of Q as determined previously. Then $r_{ij} = \vec{u}_i \cdot \vec{v}_j$, where r_{ij} is the (i, j) entry of R if $i \leq j$, 0 otherwise.
    - Property: $|\det A| = |\det R| = \left| \prod_{i=0}^{n} r_{ii} \right| = \left| \prod_{i=0}^{n} \lambda_i \right|$
    - Applications: least squares problem, finding eigenvalues
- The eigenvalues associated with distinct eigenvectors are linearly independent.
- **Eigenspace** of an eigenvalue is the dimension of its associated eigenvectors.
- **Geometric multiplicity** of an eigenvalue is the dimension of its eigenspace.
- **Algebraic multiplicity** of an eigenvalue is its multiplicity in the characteristic polynomial $\det(A - \lambda I)$
    - The geometric multiplicity of an eigenvalue is at most the algebraic multiplicity
    - A matrix is defective iff the geometric multiplicity of each eigenvalue is its algebraic multiplicity
- An $n \times n$ matrix A is **diagonalizable** if there exists an invertible matrix P and a diagonal matrix D such that $D = P^{-1}AP$.
    - A matrix is diagonalizable iff it is not defective (i.e. an $n \times n$ matrix A is diagonalizable iff it has n independent eigenvectors).
    - D is an $n \times n$ diagonal matrix with the eigenvalues of A as its diagonal entries, while P is the matrix with its corresponding independent eigenvectors as its columns.
    - $A = PDP^{-1}$
        - This is called the **eigendecomposition** or **spectral decomposition**.
        - $A^k = PD^k P^{-1}$
    - Any symmetric matrix is **orthogonally diagonalizable**, or that there exists an orthogonal matrix Q and diagonal matrix D such that $D = Q^T AQ$
        - A matrix is symmetric iff it is orthogonally diagonalizable
            - Symmetric matrices are never defective
        - $A = QDQ^T$, where D is an $n \times n$ diagonal matrix with the eigenvalues of A as its diagonal entries, while P is the matrix with its corresponding normalized eigenvectors as its columns.